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Technical Note

Thermally developing forced convection in a channel occupied by a porous medium saturated by a non-Newtonian fluid

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Abstract

The classical Graetz methodology is applied to investigate the thermal development of forced convection in a parallel plate channel filled by a saturated porous medium, with walls held at constant temperature, for the case of a non-Newtonian fluid of power-law type. A Brinkman-Forchheimer model is used for the momentum equation. The analysis for the case of small modified Darcy number leads to expressions for the local Nusselt number and average Nusselt number as functions of the dimensionless longitudinal coordinate, the power-law index, a modified Darcy number, and a modified Reynolds-Forchheimer number (with the last three parameters being involved via a boundary-layer thickness).

1. Introduction

Because of the use of hyperporous media in the cooling of electronic equipment, there has recently been renewed interest in the problem of forced convection in a porous medium channel or duct. The thermal development aspect has been treated using Graetz-type analysis in a series of papers by the present authors [1–6]. In all these papers the fluid occupying the porous medium has been Newtonian. The purpose of this paper is to extend the analysis of the thermal development to the case of a power-law fluid.

The analysis for the case of a Darcy model is trivial because the case of slug flow is covered by the classical Graetz analysis. We therefore use the Brinkman-Forchheimer extension to model the momentum equation. The fully developed situation with this model has been analyzed by Nakayama and Shenoy [7] for the case of parallel plane walls subject to uniform heat flux. It appears that no analytical treatment of the case of walls held at uniform temperature has been carried out, though a numerical treatment has been reported by Chen and Hadim [8,9].

2. Analysis

For the steady-state hydrodynamically-developed situation we have unidirectional flow in the x^*-direction

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between impermeable boundaries at $y^* = -H$ and $y^* = H$. The temperature on each boundary is held constant at the uniform value T_w^*. At $x^* = 0$ the inlet temperature T_{IN}^* is assumed constant and uniform.

Following Nakayama and Shenoy [7], the momentum equation is taken as

$$\frac{\mu}{\phi^n}\frac{d}{dy^*}\left[\left|\frac{du^*}{dy^*}\right|^{n-1}\frac{du^*}{dy^*}\right] - \left(\frac{\mu}{K}\right)u^{*^n} - \rho b u^{*^2} + G = 0 \quad (1)$$

where μ is the fluid consistency of the inelastic non-Newtonian power-law fluid (analogous to the fluid viscosity for a Newtonian fluid), ϕ is the porosity, n is the power-law index, K the permeability for such a fluid, ρ is the density of the fluid, b is a Forchheimer coefficient and G is the applied pressure gradient ($-\,dp^*/dx^*$).

We define dimensionless variables

$$\tilde{x} = \frac{x^*}{PeH}, \quad y = \frac{y^*}{H}, \quad u = \frac{u^*}{u_D^*}. \quad (2)$$

The Péclet number Pe is defined in terms of REV volume-averaged quantities as

$$Pe = \frac{\rho c_P H U}{k}. \quad (3)$$

The characteristic velocity u_D^* and the modified Reynolds number Re_K are defined by

$$u_D^* = (KG/\mu)^{1/n}, \quad (4)$$

$$Re_K = \rho b K u_D^{*\,2-n}/\mu. \quad (5)$$

The modified Darcy number Da is now defined by

$$Da = (K/\phi^n)^{2/(1+n)}/H^2. \quad (6)$$

In this analysis it is assumed than $n < 2$. (As noted by Nakayama and Shenoy [7], shear flows with $n > 2$ are not of much practical interest.)

The appropriate boundary conditions (no-slip and symmetry) in dimensionless form are

$$u = 0 \text{ at } y = 1, \quad \text{and } du/dy = 0 \quad \text{at } y = 0. \quad (7a,b)$$

The dimensionless form of Eq. (1) is

$$\frac{d}{dy}\left[\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy}\right] = \frac{u^n + Re_K u^2 - 1}{Da^{(1+n)/2}} \quad (8)$$

A simple closed form of the solution of Eq. (8) subject to the boundary conditions (7a,b) is probably not obtainable for the case of general n. Hence we seek an approximate solution. For most practical situations Da will be small compared with unity, and hence a perturbation approach is in order. There is a thin boundary layer near $y = 1$. Outside this boundary layer, the solution is given by $u = u_c$, where u_c is the centerline velocity found by solving

$$u_c^n + Re_K u_c^2 - 1 = 0. \quad (9)$$

Scale analysis reveals that the thickness of this boundary layer is δ where

$$\delta \sim Da^{1/2}/[1 + Re_K u_c^{2-n}]^{1/(1+n)} \quad (10)$$

(This is confirmed by Eq. (14) below.)

As Nakayama and Shenoy [7] have shown, it is possible to transform Eq. (8) to a boundary-layer equation for which a similarity solution is possible for the case of sufficiently small Da. However, in a pioneering study of the heat transfer problem an accurate velocity profile is not needed and so a simple integral treatment is warranted. Integration of Eq. (8) across the half channel, and the use of Eq. (7b), gives (since du/dy is negative when $y = 1$)

$$-(|du/dy|)^n|_{y=1} = Da^{-(1+n)/2}\int_0^1 (u^n + Re_K u^2 - 1)dy. \quad (11)$$

The velocity profile function is approximated by

$$u = u_c \quad \text{for } 0 \leqslant y \leqslant 1 - \delta, \quad (12)$$

$$u = u_c(1 - y)/\delta \text{ for } 1 - \delta \leqslant y \leqslant 1. \quad (13)$$

Substitution into Eq. (11) yields

$$\delta = Da^{1/2}[n/(1 + n) + (2/3)Re_K u_c^{2-n}]^{-1/(1+n)}. \quad (14)$$

The mean velocity U^* and the bulk mean temperature T_m^* are defined by

$$U^* = \frac{1}{H}\int_0^H u^* dy^*, \quad T_m^* = \frac{1}{HU^*}\int_0^H u^* T^* dy^*. \quad (15)$$

Here T^* is the volume-averaged temperature. Further dimensionless variables are defined by

$$\hat{u} = \frac{u^*}{U^*}, \quad \theta = \frac{T^* - T_w^*}{T_{IN}^* - T_w^*}. \quad (16)$$

From Eqs. (11)–(16) we deduce that

$$\hat{u} = 2/(2 - \delta) \quad \text{for } 0 \leqslant y \leqslant 1 - \delta, \quad (17)$$

$$\hat{u} = 2(1 - y)/(2\delta - \delta^2) \quad \text{for } 1 - \delta \leqslant y \leqslant 1. \quad (18)$$

The analysis of the thermal problem now follows the analysis in Nield et al. [1].

The Nusselt number Nu is defined as

$$Nu = \frac{2Hq''}{k(T_w^* - T_m^*)}. \quad (19)$$

Local thermal equilibrium is assumed. (The case of local thermal non-equilibrium could be the subject of a later investigation.) It is also assumed that the Péclet number is sufficiently large for axial conduction to be neglected. In addition we have made the usual assumptions made in studies of forced convection in porous media, e.g. uniform porosity and other properties, negligible thermal dispersion, negligible viscous dissipation. The steady-state thermal energy equation is then

$$u^*\frac{\partial T^*}{\partial x^*}=\frac{k}{\rho c_p}\frac{\partial^2 T^*}{\partial y^{*2}}. \tag{20}$$

In nondimensional form this becomes

$$\hat{u}\frac{\partial\theta}{\partial\tilde{x}}=\frac{\partial^2\theta}{\partial y^2}. \tag{21}$$

Use of the first law of thermodynamics leads to

$$\frac{\mathrm{d}T_m^*}{\mathrm{d}x^*}=\frac{q''}{\rho c_p H U^*} \tag{22}$$

where

$$q''=h(T_w^*-T_m^*).$$

Since here the wall temperature T_w^* is held uniform, it follows that

$$T_w^*-T_m^*=(T_w^*-T_{\mathrm{IN}}^*)\mathrm{e}^{-\beta\tilde{x}}, \tag{23}$$

where T_{IN}^* is the inlet temperature and the Biot number β is defined as

$$\beta=\frac{hH}{k}. \tag{24}$$

The problem now is to solve Eq. (21) subject to the conditions

$$\theta=1 \quad \text{at } \tilde{x}=0, \quad \theta=0 \quad \text{at } y=1,$$
$$\mathrm{d}\theta/\mathrm{d}y=0 \quad \text{at } y=0. \tag{25a,b,c}$$

Separation of variables, following the assumption that

$$\theta=\Xi(\tilde{x})Y(y), \tag{26}$$

leads to two linear and homogeneous equations for Ξ and Y,

$$\Xi'+\lambda^2\Xi=0, \tag{27}$$

$$Y''+\lambda^2\hat{u}Y=0. \tag{28}$$

Eq. (28) together with the boundary conditions

$$Y'(0)=Y(1)=0 \tag{29}$$

defines an eigenvalue problem of Sturm-Liouville type with eigenvalues λ_n and corresponding eigenfunctions $Y_n(y)$ for n = 1,2,3,.... In particular,

$$Y_n''+\lambda_n^2\hat{u}Y_n=0. \tag{30}$$

The required solution is the series

$$\theta=\sum_{n=1}^{\infty}C_nY_n(y)\exp(-\lambda_n^2\tilde{x}), \tag{31}$$

where the constants C_n are determined by the entry condition (25a). Since the eigenfunctions satisfy the orthogonality condition

$$\int_0^1\hat{u}Y_mY_n\mathrm{d}y=0 \quad \text{if } m\neq n \tag{32}$$

it follows that

$$C_n=\frac{\int_0^1\hat{u}Y_n\mathrm{d}y}{\int_0^1\hat{u}Y_n^2\mathrm{d}y}. \tag{33}$$

If θ_m is defined by

$$\theta_m=\frac{T_m-T_w}{T_{\mathrm{IN}}-T_w} \tag{34}$$

then it follows that

$$\theta_m=\int_0^1\hat{u}\theta\,\mathrm{d}y=\sum_{n=1}^{\infty}\frac{G_n}{\lambda_n^2}\exp(-\lambda_n^2\tilde{x}) \tag{35}$$

where

$$G_n=\int_0^1C_n\lambda_n^2\hat{u}Y_n\,\mathrm{d}y. \tag{36}$$

Eq. (19) leads to

$$Nu=\frac{-2}{\theta_m}\frac{\partial\theta}{\partial y}\bigg|_{y=1}=\frac{2\sum_{n=1}^{\infty}G_n\exp(-\lambda_n^2\tilde{x})}{\sum_{n=1}^{\infty}(G_n/\lambda_n^2)\exp(-\lambda_n^2\tilde{x})}. \tag{37}$$

This gives the local Nusselt number. The mean Nusselt number, averaged over a length $\tilde{x}$, is

$$\overline{Nu}=\frac{1}{\tilde{x}}\int_0^{\tilde{x}}Nu\,\mathrm{d}\tilde{x}=-\frac{2}{\tilde{x}}\ln\left(\frac{1}{\theta_m}\right). \tag{38}$$

In deriving the last equality use has been made of Eq. (23).

3. Results and discussion

We note that the Péclet number is involved only in the scaling of the axial coordinate. Also, the three parameters Da, Re_K and n appear in the heat-transfer analysis only in the combination δ, the boundary layer thickness. Furthermore, the centerline velocity cancels out of the heat-transfer equations except for an appearance (in conjunction with Re_K) in the expression (14) for δ, and is itself to be calculated from Eq. (9) in which Re_K and n appears. Hence it makes sense to plot the Nusselt number versus dimensionless axial coordinate for various values of δ, and supplement these plots by Table 1 facilitating the determination of $\delta/Da^{1/2}$ from the values of n and Re_K using Eqs. (9) and (14). It is envisaged that readers will use input values of n and Re_K to find the appropriate entry in Table 1, use that together with an input value of Da to obtain a value of δ, and finally use that value together with Fig. 1 to obtain the appropriate values of Nu.

Table 1 may be supplemented by the following approximate expressions. When Re_K is small, then approximately

$$\delta=Da^{1/2}\left(1+\frac{1}{n}\right)^{1/(1+n)}\left\{1-\frac{2}{3n}Re_K\right\}. \tag{39}$$

Table 1
Values of $\delta/Da^{1/2}$ for various values of n and Re_K, calculated from Eq. (14) with the use of Eq. (9)

$n\backslash Re_K$	0.001	0.01	0.1	1	10	100	1000
0.1	8.788	8.372	6.857	5.006	3.676	2.798	2.202
0.2	4.436	4.319	3.726	2.740	1.953	1.428	1.074
0.3	3.083	3.026	2.688	1.996	1.392	0.983	0.710
0.4	2.443	2.408	2.181	1.636	1.120	0.766	0.532
0.5	2.077	2.054	1.885	1.427	0.961	0.638	0.426
0.6	1.844	1.826	1.694	1.293	0.858	0.553	0.356
0.7	1.684	1.670	1.561	1.201	0.785	0.492	0.307
0.8	1.568	1.556	1.465	1.134	0.732	0.447	0.269
0.9	1.481	1.471	1.392	1.085	0.691	0.411	0.240
1.0	1.413	1.405	1.335	1.047	0.659	0.382	0.217
1.1	1.360	1.352	1.290	1.018	0.633	0.358	0.198
1.2	1.316	1.310	1.254	0.994	0.611	0.338	0.182
1.3	1.281	1.275	1.224	0.975	0.593	0.321	0.169
1.4	1.251	1.246	1.199	0.960	0.577	0.306	0.158
1.5	1.226	1.221	1.178	0.947	0.564	0.293	0.148
1.6	1.205	1.200	1.160	0.936	0.552	0.281	0.139
1.7	1.186	1.182	1.144	0.928	0.541	0.271	0.132
1.8	1.170	1.167	1.131	0.920	0.531	0.262	0.125
1.9	1.157	1.153	1.119	0.914	0.523	0.253	0.120
2.0	1.144	1.141	1.109	0.909	0.515	0.246	0.114

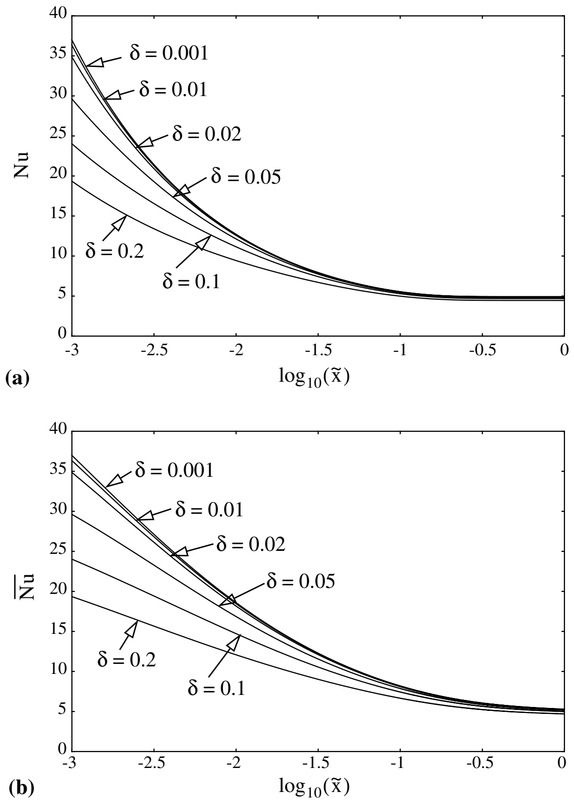

Fig. 1. Plots of (a) local Nusselt number, and (b) average Nusselt number vs. longitudinal coordinate, for various values of the boundary layer thickness.

When Re_K is large, then approximately

$$\delta = Da^{1/2}\left(\frac{3}{2}\right)^{1/(1+n)} Re_K^{-n/2(1+n)}. \tag{40}$$

The results shown in Fig. 1 are as expected. The plots for $\delta = 0.001$ approximate those for $\delta = 0$ (slug flow) and the results for Nu agree with those reported by Nield et al. [6]. It is evident that Nu decreases consistently as δ increases. As expected, the values for $\overline{Nu}$ exceed those for Nu and the values for these two quantities coincide when $\tilde{x}$ becomes large (corresponding to fully developed convection). Also as expected, the fully developed value for the case of $\delta = 0.001$ is close to $\pi^2/2 = 4.9348$, the value for slug flow.

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